Feasibility of Rotational Destratification of Space-Stored Liquid Cryogens

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A method is presented for determining the onset of a thermally induced, convective mixing motion in a model of a cylindrical space-storage tank subjected to a constant heat flux at its outer boundary. The model simulates a completely filled, liquid cryogen tank with shear-free ends rotating in a low-gravity environment. The governing disturbance equations reduce to a self-adjoint eigenvalue problem for the critical Rayleigh number (the stability criterion). Using the Rayleigh-Ritz technique of approximating eigenvalues with an equivalent variational principle, the critical Rayleigh number is calculated as a function of the Taylor number (the ratio of Coriolis forces to viscous forces) and the aspect ratio (length-to-diameter ratio) of the cylindrical tank. The results indicate that the critical Rayleigh number is a monotonically increasing function of Taylor number and a monotonically decreasing function of aspect ratio.

I. Introduction

BECAUSE of their excellent propulsive characteristics, cryogens are used as primary propellants for space vehicles. A major problem associated with unmixed space-stored cryogens is the determination of the rate of tank-pressure rise due to external heating (e.g., solar radiation). Typically a storage tank is designed with some upper limit on pressure and an associated tank-wall thickness. If the allowable pressure is increased, there is a corresponding increase in tank-wall thickness and tank weight. For short missions the tank pressure is maintained by continuous vent. However, for long-duration mission the loss of cryogen through continuous vent is excessive.

To circumvent this problem the tank pressure is allowed to build up, and its rate of buildup is controlled by employing appropriate tank-wall insulation. This method is inefficient since in a zero-gravity environment, buoyant forces are nonexistent and no convective mixing can occur. The cryogen becomes thermally stratified and the maximum heat-storage capability of the cryogen is not utilized. At present there are two upper limits that can be placed on the rate of tank-pressure rise. A maximum upper limit is obtained if it is assumed that all the heat entering the tank goes to the vapor. If it is assumed that all the heat entering the tank goes to the completely mixed liquid, then a minimum upper limit is obtained. Information from Holmes[4] was used to construct the two tank-pressure history curves represented schematically in Fig. 1. Since the thermal stratification problem is not completely understood, the tank designer must use the maximum upper limit curve for safety reasons. Thus, frequent depressurization cycles must be scheduled to insure that a space mission of long duration will be successful.

If the cryogen is completely mixed, a 200- to 300-day mission can be accomplished without depressurizing the tank. To reduce the number of depressurization cycles, pumps can be installed in the tank to circulate the liquid cryogen and maintain the cryogen at a uniform temperature. In this study we consider the possibility of rotating the tank to initiate and maintain a thermally destratifying convective motion within the vessel. Rotational destratification could supplement or substitute for the more conventional methods of mixing.

In order to examine the proposed method, the important features of the system are incorporated into a simple theoretical model of an orbiting propellant tank. A fluid-filled rotating cylinder (with angular velocity Ω) in a zero-gravity environment is subjected to a constant heat flux q at its outer boundary. A convective motion will be initiated within the cylinder when the thermally induced buoyant forces (radially directed) are sufficient to overcome viscous forces. The marginally unstable state is determined by the critical value of the Rayleigh number. The problem is thus reduced to a thermal stability problem similar to those discussed by

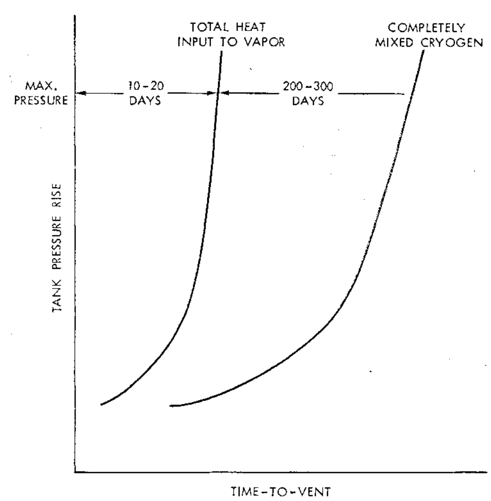

Fig. 1 Schematic representation of cryogen tank pressure history.

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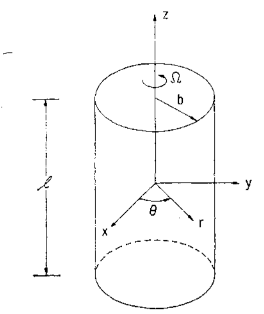

Fig. 2 The geometry and coordinate system of the cylindrical tank.

Chandrasekhar.[1] Here the Rayleigh number is given by

$$R = q\alpha\Omega^2 b^5/\nu k\kappa \tag{1}$$

where b is the cylinder radius, and α, ν, k, and κ are the fluid coefficients of thermal expansion, kinematic viscosity, thermal diffusivity, and thermal conductivity, respectively. The critical Rayleigh number is a function of a parameter which appears in all rotating fluid systems, the Taylor number,

$$T = 4\Omega^2 b^4/\nu^2 \tag{2}$$

and a geometrical shape parameter, the cylinder aspect ratio $l/2b$, where l is the cylinder length. In what follows, the functional relationship between critical Rayleigh number, Taylor number, and cylinder aspect ratio is calculated for the rotating fluid cylinder.

II. Governing Equations

Initially a quasi-incompressible (Boussinesq) fluid fills a circular cylinder which is rotating about its z axis (see Fig. 2) in a zero-gravity environment. A constant heat flux is maintained on the cylinder's radial periphery at all times. The initial velocity, temperature, and pressure distributions in the rotating frame of reference are given by

$$\left.\begin{aligned} \mathbf{u}_0 &= 0 \\ \nabla\tau_0 &= [q/(\kappa b)]r\mathbf{e}_r \\ \nabla p_0 &= \rho\Omega^2[1 - (\alpha q/2\kappa b)r^2]r\mathbf{e}_r \end{aligned}\right\} \tag{3}$$

where ρ is the mean fluid density, r is the radial coordinate, and $\mathbf{e}_r$ is a unit radius vector.

It is assumed that instability appears via a marginal stationary state; hence the governing perturbation equations in dimensionless form are

$$\operatorname{div}\mathbf{u} = 0 \tag{4}$$

$$\operatorname{curl}\operatorname{curl}\mathbf{u} + \operatorname{grad} p + R\tau r\mathbf{e}_r + T^{1/2}\mathbf{u}\times\mathbf{k} = 0 \tag{5}$$

$$\nabla^2\tau = \mathbf{u}\cdot r\mathbf{e}_r \tag{6}$$

where $\mathbf{u}$, τ, and p are the velocity, temperature, and pressure disturbances measured in units of k/b, qb/κ, and $\rho\nu k/b^2$, respectively. Here $\mathbf{k}$ is the unit vector along the z axis. If it is assumed that all perturbations are axially symmetric (independent of the azimuth angle, θ), then it is convenient to express the solenoidal velocity disturbance as the superposition of a poloidal and a toroidal vector field. Thus one can write

$$\mathbf{u} = \operatorname{curl}[\Psi(r,z)\mathbf{e}_\theta] + T^{1/2}\Phi(r,z)\mathbf{e}_\theta \tag{7}$$

where Ψ and Φ are defining scalars, and $\mathbf{e}_\theta$ is a unit vector in the azimuthal direction. After eliminating the pressure by taking the curl of Eq. (5), and using the velocity field description given by Eq. (7), the governing perturbation equations assume the form

$$\operatorname{curl}\operatorname{curl}\operatorname{curl}\operatorname{curl}(\Psi\mathbf{e}_\theta) + R\operatorname{curl}(r\tau\mathbf{e}_r) - T\operatorname{curl}(\Phi\mathbf{e}_\theta\times\mathbf{k}) = 0 \tag{8}$$

$$\operatorname{curl}\operatorname{curl}(\Phi\mathbf{e}_\theta) + \mathbf{k}X\operatorname{curl}(\Psi\mathbf{e}_\theta) = 0 \tag{9}$$

$$\nabla^2\tau = r\mathbf{e}_r\cdot\operatorname{curl}(\Psi\mathbf{e}_\theta) \tag{10}$$

Equations (8–10) may be written in an equivalent form by carrying out the indicated vector operations. One then obtains

$$L^4\Psi + Rr(\partial\tau/\partial z) - T(\partial\Phi/\partial z) = 0 \tag{11}$$

$$L^2\Phi = -(\partial\psi/\partial z) \tag{12}$$

$$\nabla^2\tau = -r(\partial\psi/\partial z) \tag{13}$$

where

$$L^2 \equiv \partial/\partial r[\partial/\partial r + 1/r] + \partial^2/\partial z^2$$

On the constant heat flux, rigid, radial bounding surface, the boundary conditions are

$$\Psi = (\partial\psi/\partial r) = \Phi = (\partial\tau/\partial r) = 0 \text{ at } r = 1 \tag{14}$$

For convenience, the cylinder's end bounding surfaces ($z = \pm l/2b$) are assumed to be insulated, shear-free boundaries; hence the boundary conditions on these surfaces are

$$\psi = (\partial^2\psi/\partial z^2) = (\partial\Phi/\partial z) = (\partial\tau/\partial z) = 0 \text{ at } z = \pm(l/2b) \tag{15}$$

Equations (11–13) and the boundary conditions given by Eqs. (14) and (15) form a self-adjoint eigenvalue problem for the Rayleigh number with the Taylor number as a parameter. A variational principle corresponding to the eigenvalue problem can be constructed as follows. Multiply Eqs. (11), (12), and (13) by Ψ, Φ, and τ, respectively, and then integrate over the cylindrical region (indicated by $\langle\ \rangle$). Integrating by parts and employing the boundary conditions yields

$$\langle\psi L^4\psi\rangle + R\langle r(\partial\tau/\partial z)\psi\rangle - T\langle(\partial\Phi/\partial z)\psi\rangle = 0$$

$$\langle\Phi L^2\Phi\rangle = \langle\psi(\partial\Phi/\partial z)\rangle$$

$$\langle\tau\nabla^2\tau\rangle = \langle r(\partial\tau/\partial z)\psi\rangle$$

Combining these expressions yields the Rayleigh number as the extremal values of

$$R = [\langle\Psi L^4\Psi\rangle - T\langle\Phi L^2\Phi\rangle]/-\langle\tau\nabla^2\tau\rangle \tag{16}$$

with

$$L^2\Phi = -(\partial\psi/\partial z) \tag{17}$$

$$\nabla^2\tau = -r(\partial\psi/\partial z) \tag{18}$$

in the cylindrical region and the boundary conditions given by Eqs. (14) and (15). The Rayleigh-Ritz approximation method is used with the preceding variational principle to obtain upper-bound estimates to the critical Rayleigh number. A trial function for Ψ is constructed which satisfies all the boundary conditions, and includes some variational parameters. Equations (17) and (18) are then solved for Φ and τ subject to their proper boundary conditions. After specifying the Taylor number, the Rayleigh number associated with the trial function can be found from Eq. (16) and minimized with respect to the parameters. The accuracy of the approximation depends on the nature of the trial function construction.

III. Approximate Solution Method

First the trial function sequence Ψ is constructed from a complete set of coordinate functions, each of which satisfies the boundary conditions on Ψ. It is convenient to choose this set of functions so that Eqs. (17) and (18) can be solved with little effort and the integrals appearing in Eq. (16) can be evaluated analytically. Thus Ψ is chosen as

$$\Psi = \sum_{j=1}^{\infty} A_j\Psi_j \tag{19}$$

where A_j are the variational parameters,

$$\Psi_j = f_j(r) \cos az$$

and

$$f_j(r) \equiv [J_1(\alpha_j r)/J_1(\alpha_j)] - [I_1(\alpha_j r)/I_1(\alpha_j)] \quad (20)$$

Here $a = \pi b/l$, α_j is the jth root of

$$J_1(\alpha)I_2(\alpha) + J_2(\alpha)I_1(\alpha) = 0 \quad (21)$$

J_n is the Bessel function of the first kind, and I_n is the modified Bessel function of the first kind. Using the aforementioned expression for ψ and Eq. (17), one can obtain

$$L^2\Phi = a \sum_{j=1}^{\infty} A_j f_j(r) \sin az \quad (22)$$

and

$$\Phi = \sum_{j=1}^{\infty} A_j \Phi_j \quad (23)$$

where

$$\Phi_j = a \left\{ \frac{2\alpha_j^2}{\alpha_j^4 - a^4} \frac{I_1(ar)}{I_1(a)} - \frac{1}{\alpha_j^2 + a^2} \frac{J_1(\alpha_j r)}{J_1(\alpha_j)} - \frac{1}{\alpha_j^2 - a^2} \frac{I_1(\alpha_j r)}{I_1(\alpha_j)} \right\} \sin az \quad (24)$$

Substituting Ψ into Eq. (18) yields

$$\nabla^2 \tau = ar \sum_{j=1}^{\infty} A_j f_j(r) \sin az \quad (25)$$

Equation (25) is then solved for τ using the method of variation of parameters. The result is

$$\tau = \sum_{j=1}^{\infty} A_j \tau_j \quad (26)$$

where

$$\tau_j = a^2 \left\{ I_0(ar) \int_0^r r^2 f_j(r) K_0(ar) dr + \frac{K_1(a)}{I_1(a)} I_0(ar) \times \int_0^1 r^2 f_j(r) I_0(ar) dr - K_0(ar) \int_0^r r^2 f_j(r) I_0(ar) dr \right\} \quad (27)$$

and K_n is the modified Bessel function of the second kind. Substituting the expressions for Ψ, Φ, and τ into Eq. (16), and minimizing with respect to the variational parameters, leads to the following infinite system of linear homogeneous equations for A_j:

$$\sum_{j=1}^{\infty} A_j \left[\frac{\langle \Psi_j L^4 \Psi_k \rangle - T \langle \Phi_j L^2 \Phi_k \rangle}{R} + \langle \tau_j \nabla^2 \tau_k \rangle \right] = 0 \quad (28)$$

where the following elements of the coefficient matrix are calculated in the Appendix:

$$\langle \Psi_j L^4 \Psi_k \rangle = \alpha_j^4 + a^4 - 2a^2\alpha_j^2 \frac{J_0(\alpha_j)J_2(\alpha_j)}{J_1^2(\alpha_j)}, j = k$$

$$= \frac{8a^2\alpha_j^2\alpha_k^2}{\alpha_k^4 - \alpha_j^4} \left\{ \alpha_j \frac{J_0(\alpha_j)}{J_1(\alpha_j)} - \alpha_k \frac{J_0(\alpha_k)}{J_1(\alpha_k)} \right\}, j \neq k$$

$$\langle \Phi_j L^2 \Phi_k \rangle = \frac{a^4}{\alpha_j^4 - a^4} + \frac{4a^2\alpha_j^4}{\alpha_j^4 - a^{42}} a \frac{I_0(a)}{I_1(a)} - \alpha_j \frac{J_0(\alpha_j)}{J_1(\alpha_j)} + \frac{a^2\alpha_j^2}{\alpha_j^4 - a^4} \frac{J_0(\alpha_j)J_2(\alpha_j)}{J_1^2(\alpha_j)}, j = k$$

$$= 4a^2\alpha_j^2\alpha_k^2 \left\{ \frac{a}{\alpha_j^4 - a^4\alpha_k^4 - a^4} \frac{I_0(a)}{I_1(a)} + \frac{\alpha_j}{\alpha_k^4 - \alpha_j^4 a^4 - \alpha_j^4} \frac{J_0(\alpha_j)}{J_1(\alpha_j)} + \frac{\alpha_k}{\alpha_j^4 - \alpha_k^4 a^4 - \alpha_k^4} \frac{J_0(\alpha_k)}{J_0(\alpha_k)} \right\}, j \neq k \quad (29)$$

The element $\tau_j \nabla^2 \tau_k$ cannot be expressed in any simple analytical form; hence it is computed numerically. This calculated element and the aforementioned two elements are symmetric with respect to j and k. The characteristic determinant of the system of equations is

$$\| [\langle \Psi_j L^4 \Psi_k \rangle - T \langle \Phi_j L^2 \Phi_k \rangle]/R + \langle \tau_j \nabla^2 \tau_k \rangle \| = 0 \quad (30)$$

For a given Taylor number and cylinder aspect ratio, the smallest positive root of Eq. (30) is the critical Rayleigh number associated with the first mode of instability. If the trial function sequence, Eq. (19), is truncated after a finite number of terms, then a characteristic determinant with a finite number of elements is obtained. Increasing the number of coordinate functions in the approximating sequence increases the accuracy of the upper-bound estimate to the critical Rayleigh number.

The following criterion is used to decide how many coordinate functions should be included in the trial function sequence. For a given Taylor number and cylinder aspect ratio, if the critical Rayleigh number associated with the $(n + 1)$-term sequence is less than 1% smaller than the critical Rayleigh number associated with the n-term sequence, then the characteristic determinant is truncated with $(n + 1)^2$ elements. It is found for cylinder aspect ratios of interest, $0.25 < l/2b < 4.0$, and for Taylor numbers less than 10^{10}, a 10-term sequence always satisfies the aforementioned criterion. Thus the first ten roots of Eq. (21) are needed for the calculations. They are presented in Table 1.

Table 1 Roots of Eq. (21), α_j

α_1	4.6108999
α_2	7.7992738
α_3	10.958067
α_4	14.108628
α_5	17.255727
α_6	20.401045
α_7	23.545322
α_8	26.688955
α_9	29.832137
α_{10}	32.975015

The computed critical Rayleigh number approximations for various Taylor numbers and cylinder aspect ratios are presented in Table 2. These results are graphically illustrated in Fig. 3 for Taylor numbers up to 10^{13}. The values of R_c for $T > 10^{10}$ are extrapolated values. It can be seen that for large values of the Taylor number, the critical Rayleigh number is a linear function of the Taylor number. One observes that R_c/T is a monotonically increasing function of the Taylor number and approaches an asymptotic limit.

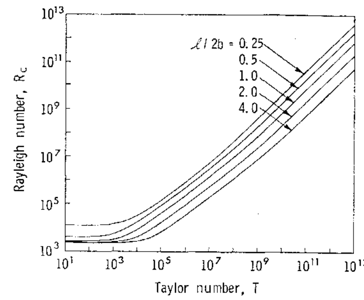

Fig. 3 The relation between critical Rayleigh and Taylor numbers for various aspect ratios.

Table 2 Critical Rayleigh number approximations

Taylor number	0.25	0.50	1.00	2.00	4.00
10^0	1.2030×10^4	3.9241×10^3	2.6329×10^3	2.3650×10^3	2.3018×10^3
10^1	1.2052×10^4	3.9391×10^3	2.6388×10^3	2.3667×10^3	2.3022×10^3
10^2	1.2265×10^4	4.0886×10^3	2.6972×10^3	2.3835×10^3	2.3066×10^3
10^3	1.4360×10^4	5.5512×10^3	3.2752×10^3	2.5509×10^3	2.3501×10^3
10^4	3.2359×10^4	1.7766×10^4	8.5341×10^3	4.1731×10^3	2.7820×10^3
10^5	1.4833×10^5	8.8904×10^4	4.2656×10^4	1.7113×10^4	6.7770×10^3
10^6	8.5886×10^5	5.0319×10^5	2.3878×10^5	9.2694×10^4	3.3984×10^4
10^7	5.5396×10^6	3.1047×10^6	1.4478×10^6	5.5015×10^5	1.9281×10^5
10^8	4.3031×10^7	2.1754×10^7	9.5190×10^6	3.4878×10^6	1.1879×10^6
10^9	4.0501×10^8	1.8882×10^8	7.3645×10^7	2.4408×10^7	7.7683×10^6
10^{10}	4.0324×10^9	1.7733×10^9	6.8208×10^8	2.0714×10^8	5.8368×10^7

From Eqs. (1) and (2), one has

$$R_c/T = \alpha\nu bq/4k\kappa \tag{31}$$

Since $\alpha\nu/4k\kappa$ is a function of fluid properties only, there is a limit on the heat flux-radius product below which no convective flow can occur at any rate of rotation. Figure 4 shows this lower limit as a function of storage tank dimensions and cryogen properties.

IV. Discussion of Results

Several approximations have been incorporated into the model used to examine the feasibility of rotating a cryogenic storage tank to initiate a convective mixing motion within the liquid cryogen. The cylindrical vessel is assumed to be completely filled. This is almost never true (except for supercritical storage) since an ullage must be available to permit thermal expansion of the liquid. When a partially filled vessel is rotating, the fluid will arrange itself so that the ullage will be along the axis of rotation. The physical problem is really that of fluid contained between rotating concentric cylinders. The outer cylinder is rigid and the inner cylinder is a shear-free boundary. This problem is not more difficult in principle; however, the mechanics of solving it are more involved. In the similar problem of the stability of couette flow,[1] the stability criterion increases as the annular gap width decreases. The same qualitative result is expected in the rotational destratification problem.

A second approximation incorporated into the model is to assume the cylindrical tank end walls to be shear free. This approximation is made by Catton and Edwards[2] for a somewhat different physical problem, but one with essentially similar mathematical characteristics. The results indicate that, if the end walls are rigid, the additional viscous dissipation that must be overcome is inconsequential when the tank aspect ratio is greater than 2.0. At a tank aspect ratio of 1.0, the theoretically predicted stability criterion is 10% lower than the experimentally observed value. Even at low aspect ratios the qualitative features of the solution are valid and the feasibility of initiating a convective flow remains.

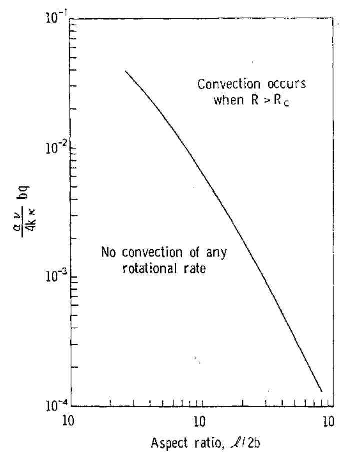

Fig. 4 The lower limit on the heat flux radius product.

Since the possibility of inducing a convective flow in a heated, spinning cylinder has been demonstrated theoretically, it is useful to consider a specific cryogen that is used in space vehicles, i.e., liquid hydrogen. The thermodynamic properties of liquid hydrogen at 37°R given by Johnson[3] are presented in Table 3. Using these properties and Fig. 3, a relation among heat flux, rate of rotation, and aspect ratio for a liquid hydrogen tank can be developed as shown in Fig. 5.

As an example of the application of these results, consider a super-insulated liquid hydrogen tank with a heat flux of 0.1 Btu hr^{-1} ft^{-2}, a 15-ft radius, and a 60-ft length. It is found from Fig. 5 that a rotational rate of 10^{-5} sec^{-1} (one revolution every 175 hr) will initiate a convective flow within the tank. This rate of rotation is extremely low and should be easy to achieve. For typical heat fluxes, tank dimensions, and cryogens, the spin rates required to initiate rotational destratification fall within typical ACS (attitude control system) operating regimes.

This memorandum has considered only the feasibility of inducing a mixing motion. The tank would have to be rotated faster than its critical speed to achieve a specified mixing rate. Experience with similar problems indicates that increasing the rate of rotation one order of magnitude above the critical speed will increase the effective thermal conductivity of the fluid by a factor of five. Further increases in spin rates will increase the effective thermal conductivity by approximately the square root of the spin rate. Since the ability of the fluid to transfer momentum (i.e., to mix) is equivalent to the ability of the fluid to transfer heat, mixing rates should depend on spin rates in a similar manner. It is not known at this point how to put a figure of merit on mixing rate. Even if it is required that the spin rate be several orders of magnitude above the critical spin rate to achieve adequate mixing, the spin rates will only be of the order of one revolution per day.

Examination of Fig. 5 reveals an interesting paradox. It can be shown for the foregoing example that if the heat flux is reduced to less than 0.035 Btu hr^{-1} ft^{-2}, convective mixing will not occur at any spin rate because the centrifugal body forces cannot overcome the viscous and Coriolis forces.

Table 3 Properties of liquid hydrogen at 37°R

Property	Value
α	9.0×10^{-3} °F^{-1}
ν	1.94×10^{-6} ft^2 sec^{-1}
k	1.71×10^{-6} ft^2 sec^{-1}
κ	6.7×10^{-2} Btu hr^{-1} ft^{-1} °F^{-1}
ρ	4.35 lbm ft^{-3}
c_p	2.5 Btu lbm^{-1} °F^{-1}

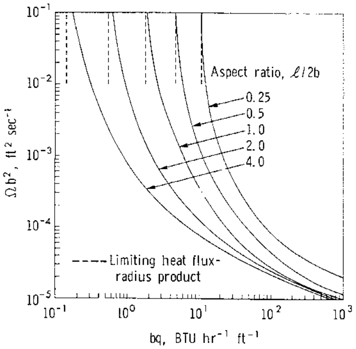

Fig. 5 Rotation-heat flux relation in liquid hydrogen for various aspect ratios.

The paradox suggests that it might be possible to optimize a particular mission by removing tank-wall insulation in order to ensure adequate mixing and thermal destratification. The average temperature of the liquid-vapor mixture would rise, but the tank vapor pressure would fall. A study of the tradeoffs among insulation thickness (weight), heat flux, boiloff liquid cryogen weight, ACS weight, etc. for various missions would be useful to determine the optimum set of conditions.

There is a need to study ways of starting, maintaining, and stopping rotation of the large tanks used in long-duration space missions. The example tank is filled with almost 200,000 lb of liquid hydrogen. Add to this an equivalent tank weight, and it is clear that even the maintenance of small spin rates would require large amounts of energy. The start-up problem requires a study of the time it takes the cryogen to spin-up to the tank's design speed. Since the kinematic viscosity of cryogens is low, the spin-up time may be long and the tank designer may need to put baffles inside the tank to achieve rapid spin-up. Once the tank designer decides on an operating spin rate, he must examine how this spin-rate influences the ullage and subsequent sloshing and surface tension effects. These are nonlinear problems that are beyond the scope of this feasibility study.

Appendix: Evaluation of Matrix Elements

It is convenient to note the following formulas before evaluating the elements of the coefficient matrix given in Eq. (28). Let $f_j(r)$ be defined by Eq. (20) and

$$g_j(r) \equiv J_1(\alpha_j r)/J_1(\alpha_j) + I_1(\alpha_j r)/I_1(\alpha_j) \tag{A1}$$

One can easily verify

$$\langle f_j f_k \rangle = \delta_{jk} \tag{A2}$$

$$\begin{aligned} \langle g_j f_k \rangle &= -\frac{J_0(\alpha_j)J_2(\alpha_j)}{J_1^2(\alpha_j)}, \; j = k \\ &= \frac{4\alpha_k^2}{\alpha_k^4 - j^4} \left\{ \alpha_j \frac{J_0(\alpha_j)}{J_1(\alpha_j)} - \alpha_k \frac{J_0(\alpha_k)}{J_1(\alpha_k)} \right\}, \; j \neq k \end{aligned} \tag{A3}$$

Since $\Psi_j = f_j \cos az$, then it can be shown that

$$L^4\Psi_j = \{(\alpha_j^4 + a^4)f_j + 2a^2\alpha_j^2 g_j\} \cos az \tag{A4}$$

Hence, using Eqs. (A2) and (A3),

$$\langle \Psi_j L^4 \Psi_k \rangle = \langle \{(\alpha_k^4 + a^4) f_j f_k + 2a^2\alpha_k^2 g_k f_j\} \cos^2 az \rangle$$

and

$$\begin{aligned} \langle \Psi_j L^4 \Psi_k \rangle &= \alpha_j^4 + a^4 - 2a^2\alpha_j^2 \frac{J_0(\alpha_j)J_2(\alpha_j)}{J_1^2(\alpha_j)}, \; j = k \\ &= \frac{8a^2\alpha_k^2\alpha_j^2}{\alpha_k^4 - \alpha_j^4} \left\{ \alpha_j \frac{J_0(\alpha_j)}{J_1(\alpha_j)} - \alpha_k \frac{J_0(\alpha_k)}{J_1(\alpha_k)} \right\}, \; j \neq k \end{aligned} \tag{A5}$$

One obtains from Eq. (22), $L^2\Phi_k = af_k \sin az$, and observes that Φ_j [see Eq. (24)] can be rearranged as

$$\Phi_j = a \left\{ \frac{2\alpha_j^2}{\alpha_j^4 - a^4} \frac{I_1(ar)}{I_1(a)} - \frac{\alpha_j^2}{\alpha_j^4 - a^4} g_j + \frac{a^2}{\alpha_j^4 - a^4} f_j \right\} \sin az \tag{A6}$$

Hence, using Eqs. (A2) and (A3),

$$\begin{aligned} \langle \Phi_j L^2 \Phi_k \rangle &= a^2 \left\langle \left\{ \frac{2\alpha_j^2}{\alpha_j^4 - a^4} \frac{I_1(ar)}{I_1(a)} f_k - \frac{\alpha_j^2}{\alpha_j^4 - a^4} g_j f_k + \frac{a^2}{\alpha_j^4 - a^4} f_j f_k \right\} \sin^2 az \right\rangle \\ &= a^2 \left\{ \frac{2\alpha_j^2}{\alpha_j^4 - a^4} \left[\frac{2\alpha_k^2}{\alpha_k^4 - a^4} a \frac{I_0(a)}{I_1(a)} - \frac{2\alpha_k^2}{\alpha_k^4 - a^4} \alpha_k \frac{J_0(\alpha_k)}{J_1(\alpha_k)} \right] - \frac{\alpha_j^2}{\alpha_j^4 - a^4} \langle g_j f_k \sin^2 az \rangle + \frac{a^2}{\alpha_j^4 - a^4} \langle f_j f_k \sin^2 az \rangle \right\} \end{aligned}$$

and

$$\begin{aligned} \langle \Phi_j L^2 \Phi_k \rangle &= \frac{4a^2\alpha_j^4}{(\alpha_j^4 - a^4)^2} a \left\{ \frac{I_0(a)}{I_1(a)} - a_j \frac{J_0(\alpha_j)}{J_1(\alpha_j)} \right\} + \frac{a^2\alpha_j^2}{\alpha_j^4 - a^4} \frac{J_0(\alpha_j)J_2(\alpha_j)}{J_1^2(\alpha_j)} + \frac{a^4}{\alpha_j^4 - a^4}, \; j = k \\ &= 4a^2\alpha_j^2\alpha_k^2 \left\{ \frac{a}{\alpha_j^4 - a^4 \alpha_k^4 - a^4} \frac{I_0(a)}{I_1(a)} + \frac{\alpha_j}{\alpha_k^4 - \alpha_j^4 a^4 - \alpha_j^4} \frac{J_0(\alpha_j)}{J_1(\alpha_j)} + \frac{\alpha_k}{\alpha_j^4 - \alpha_k^4 a^4 - \alpha_k^4} \frac{J_0(\alpha_k)}{J_1(\alpha_k)} \right\} \\ &\qquad j \neq k \end{aligned} \tag{A7}$$

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